**ECE 830 Fall 2011 Statistical Signal Processing**

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Lecture 8: Signal Detection and Noise Assumption

# 1 Signal Detection

$$\begin{aligned} H_0: X &= W \\ H_1: X &= S + W \end{aligned}$$

where $W \sim N(0, \sigma^2 I_{n\times n})$ and $S = [s_1, s_2, \ldots, s_n]^T$ is the known signal waveform.

$$\begin{aligned} P_0(X) &= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} exp(-\frac{1}{2\sigma^2} X^T X) \\ P_1(X) &= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} exp[-\frac{1}{2\sigma^2}(X-S)^T(X-S)] \end{aligned}$$

The second equation holds because under hypothesis $H_0$, $W = X - S$.

The Log Likelihood Ratio test is

$$\log \Lambda(x) = \log \frac{P_W(X)}{P_W(X-S)} = -\frac{1}{2\sigma^2}[(X-S)^T(X-S) - X^T X] = -\frac{1}{2\sigma^2}[-2X^T + S^T S] \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'$$

After simplifying it, we can get

$$X^T S \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \gamma' + \frac{S^T S}{2} = \gamma$$

In this case, $X^T S$ is the sufficient statistics $t(x)$ for the parameter $\theta = 0, 1$. Note that $S^T S = \|S\|_2^2$ is the signal energy. The LR detector "filters" data by projecting them onto signal subspace.

## 1.1 Example 1

Suppose we want to control the probability of false alarm. For example, choose $\gamma$ so that $\mathbb{P}(X^T S > \gamma \mid H_0) \leq 0.05$.

The test statistic $X^T S$ is usually called "matched filter".

In particular, projection onto subspace spanned by $S$ is

$$P_S = \frac{SS^T}{S^T S} = \frac{S}{\|S\|} \cdot \frac{S^T}{\|S\|}$$

$$P_S X = \frac{SS^T}{\|S\|^2} X = (X^T S)\frac{S}{\|S\|^2}$$

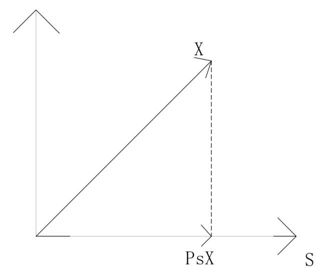


Figure 1: Projection of X onto subspace S

where $\frac{X^T S}{\|S\|^2}$ is just a number.

Geometrically, suppose the horizontal line is the subspace S and X is some other vector. The projection of vector X into subspace S can be expressed in the figure 1.

### 1.2 Example 2

Suppose the signal value $S_k$ is sinusoid.

$$S_k = \cos(2\pi f_0 k + \theta), k = 1, \ldots, n$$

The match filter in this case is to compute the value in the specific frequency. So $P_S$ in this example is a bandpass filter.

### 1.3 Performance Analysis

Next problem what we want to know is what's the probability density of $X^T S$, which is the sufficient statistics of this test.

$$\begin{aligned} H_0: X &\sim N(0, \sigma^2 I) \\ H_1: X &\sim N(S, \sigma^2 I) \end{aligned}$$

$X^T S = \sum_{k=1}^{n} X_k S_k$ is also Gaussian distributed. Recall if $X \sim N(\mu, \Sigma)$, then $Y = AX \sim N(A\mu, A\Sigma A^T)$, where A is a matrix.

Since $Y = X^T S = S^T X$, Y is a scalar. So we can get

$$\begin{aligned} H_0: X^T S &\sim N(0^T S, S^T \sigma^2 I S) = N(0, \sigma^2 \|S\|^2) \\ H_1: X^T S &\sim N(S^T S, S^T \sigma^2 I S) = N(\|S\|^2, \sigma^2 \|S\|^2) \end{aligned}$$

The probability of false alarm is $P_{FA} = Q(\frac{\gamma - 0}{\sigma\|S\|})$, and the probability of detection is $P_D = Q(\frac{\gamma - \|S\|^2}{\sigma\|S\|}) = Q(\frac{\gamma}{\sigma\|S\|} - \frac{\|S\|}{\sigma})$. Since Q function is invertible, we can get $\frac{\gamma}{\sigma\|S\|} = Q^{-1}(P_{FA})$. Therefore, $P_D = Q(Q^{-1}(P_{FA}) - \frac{\|S\|}{\sigma})$. In the equation, $\frac{\|S\|}{\sigma}$ is the square root of Signal Noise Ratio($\sqrt{SNR}$).

## 2 AWGN Assumption

Is real-world noise really additive, white and Gaussian? Well, here are a few observations. Noise in many applications (e.g. communication and radar) arose from several independent sources, all adding together

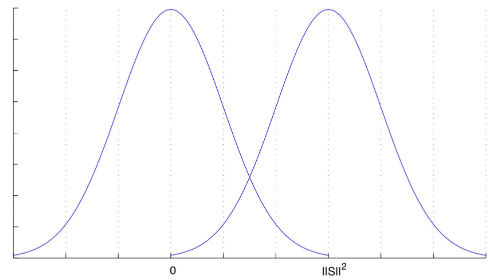


Figure 2: Distribution of $P_0$ and $P_1$

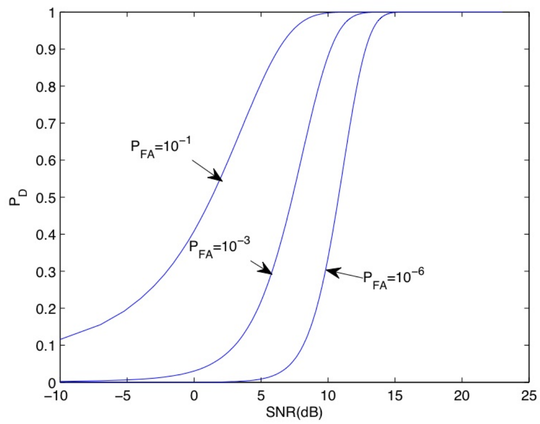


Figure 3: Relation between probability of detection and false alarm

at sensors and combining additively to the measurement. AWGN is gaussian distributed as the following formula.

$$W \sim N(0, \sigma^2 I)$$

**CLT(Central Limit Theorem):** If $x_1, \dots, x_n$ are independent random variables with means $\mu_i$ and variances $\sigma_i^2 < \infty$ ,then $Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{x_i - \mu_i}{\sigma_i} \to N(0,1)$ in distribution quite quickly.

Thus, it is quite reasonable to model noise as additive and Gaussian list in many applications. However, whiteness is not always a good assumption.

### 2.1 Example 3

Suppose $W = S_1 + S_2 + \cdots + S_k$, where $S_1, S_2, \dots S_k$ are inteferencing signals that are not of interest. But each of them is structured/correlated in time. Therefore, we need a more generalized form of noise, which is "Colored Gaussian Noise".

## 3 Colored Gaussian Noise

$W \sim N(0, \Sigma)$ is called correlated or "colored" noise, where $\Sigma$ is a structured covariance matrix.

Consider the binary hypothesis test in this case.

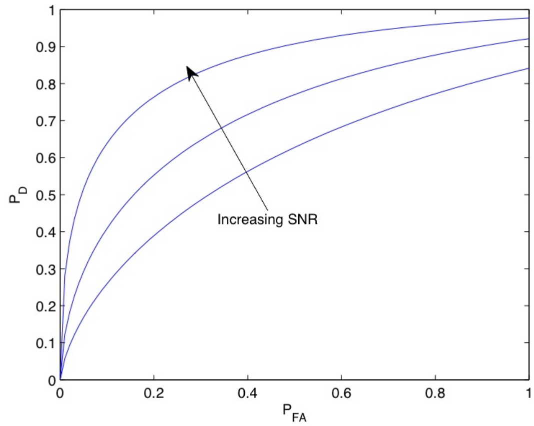


Figure 4: Relation between probability of detection and SNR

$$\begin{aligned} H_0: \; X &= S_0 + W \\ H_1: \; X &= S_1 + W \end{aligned}$$

where $W \sim N(0, \Sigma)$ and $S_0$ and $S_1$ are know signal waveforms. So we can rewrite the hypothesis as

$$\begin{aligned} H_0: \; X \sim N(S_0, \Sigma) \\ H_1: \; X \sim N(S_1, \Sigma) \end{aligned}$$

The probability density of each hypothesis is

$$P_i(X) = \frac{1}{(2\pi)^{\frac{2}{n}}(\Sigma)^{\frac{1}{2}}} exp[-\frac{1}{2}(X - S_i)^T \Sigma^{-1}(X - S_i)], i = 0, 1$$

The log likelihood ratio is

$$\log(\frac{P_1(X)}{P_2(X)}) = -\frac{1}{2}[(X-S_1)^T \Sigma (X-S_1) - (X-S_0)^T \Sigma^{-1}(X-S_0)] = X^T \Sigma^{-1}(S_1 - S_0) - \frac{1}{2} S_1^T \Sigma^{-1} S_1 + \frac{1}{2} S_0^T \Sigma^{-1} S_0 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'$$

$$(S_1 - S_0)\Sigma^{-1} X \underset{H_0}{\overset{H_1}{\gtrless}} \gamma' + \frac{S_1^T \Sigma^{-1} S_1}{2} - \frac{S_0^T \Sigma^{-1} S_0}{2} = \gamma$$

Let $t(X) = (S_1 - S_0)\Sigma^{-1} X$, we can get

$$\begin{aligned} H_0: \; t \sim N((S_1 - S_0)\Sigma^{-1} S_0, (S_1 - S_0)^T \Sigma^{-1}(S_1 - S_0)) \\ H_1: \; t \sim N((S_1 - S_0)\Sigma^{-1} S_1, (S_1 - S_0)^T \Sigma^{-1}(S_1 - S_0)) \end{aligned}$$

The probability of false alarm is

$$P_{FA} = Q(\frac{\gamma - (S_1 - S_0)^T \Sigma^{-1} S_0}{[(S_1 - S_0)^T \Sigma^{-1}(S_1 - S_0)]^{\frac{1}{2}}})$$

In this case it is natural to define

$$SNR = (S_1 - S_0)^T \Sigma^{-1}(S_1 - S_0)$$

## 3.1 Example 4

$S_1 = [\frac{1}{2}, \frac{1}{2}]$, $S_0 = [-\frac{1}{2}, -\frac{1}{2}]$, $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, $\Sigma^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$.

The test statistics is

$$y = (S_1 - S_0)\Sigma^{-1} X = [1, 1] \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{1+\rho}(x_1 + x_2)$$

$$H_0 : \ y \sim N(-\frac{1}{1+\rho}, \frac{2}{1+\rho})$$
$$H_1 : \ y \sim N(+\frac{1}{1+\rho}, \frac{2}{1+\rho})$$

The probability of false alarm is

$$P_{FA} = Q(\frac{\gamma + \frac{1}{1+\rho}}{\sqrt{\frac{2}{1+\rho}}})$$

The probability of detection is

$$P_D = Q(\frac{\gamma - \frac{1}{1+\rho}}{\sqrt{\frac{2}{1+\rho}}})$$

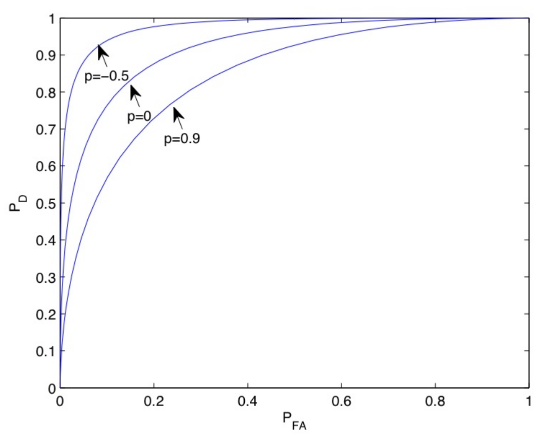


Figure 5: ROC curve at different $\rho$